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**COMMENT**

# Comment on 'Sum rules for quantum billiards'

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**Abstract.** For a $d$-dimensional rectangular box, the function $\zeta(T; s)$ considered by Itzykson *et al* is related to Epstein's zeta function and can be written in a form which exhibits the analytic structure explicitly.

Recently, Itzykson *et al* (1986) have considered the behaviour of the function

$$\zeta(T; s) \equiv \sum_{s=1}^{\infty} E_p^{-s} \tag{1}$$

about $s=1$, where $E_p$ are the eigenvalues of the Laplacian operator on the bounded domain $T$ with Dirichlet boundary conditions (which ensures $E_0 > 0$). In this comment I show that, for a $d$-dimensional rectangular box, an explicit expression for $\zeta(T; s)$ can be written which shows its analytic behaviour, and that, for a square, the behaviour of $\zeta(T; s)$ can be found from previously known results. Furthermore, explicit solutions for many other rectangles and triangles with rational side ratios are suggested by the extensive work of Glasser and Zucker (1980 and references therein).

For a $d$-dimensional box of sizes $L_1, L_2, \ldots, L_d$, $E$ are given by

$$E(n_1, \ldots, n_d) = \pi^2(n_1^2/L_1^2 + \ldots + n_d^2/L_d^2) \tag{2}$$

with $n_i = 0, 1, 2, \ldots$. We make use of Epstein's zeta function (see Glasser and Zucker 1980) written in the form (Ziff *et al* 1977)

$$C_s(\omega_1, \ldots, \omega_d) \equiv \Gamma(s)\pi^{-s} \sum_{n_i}{}' (N^2)^{-s} \qquad (s > d/2) \tag{3}$$

where $N^2 \equiv n_1^2/\omega_1^2 + \ldots + n_d^2/\omega_d^2$, $\omega_1\omega_2 \ldots \omega_d = 1$, and the sum is over all $n_i$ except the single term $n_1 = 0$, $n_2 = 0 \ldots$ (which is denoted by the prime). $C_s$ satisfies

$$C_s(\omega_1, \omega_2, \ldots, \omega_d) = 1/(s - d/2) - 1/s + \sum_{n_i}{}' E_{1-s}(\pi N^2) + \sum_{n_i}{}' E_{s+1-d/2}(\pi \tilde{N}^2) \tag{4}$$

where $\tilde{N}^2 \equiv n_1^2\omega_1^2 + \ldots + n_d^2\omega_d^2$ and $E_n(s) \equiv \int_1^\infty t^{-n} e^{-st}\, dt$. Equation (4) provides the analytic continuation for $C_s$ for all $s$, implies the identity $C_s(\omega_1, \ldots, \omega_d) = C_{d/2-s}(1/\omega_1, \ldots, 1/\omega_d)$ and shows that $C_s$ has only two poles, at $z = 0$ and $z = d/2$. For $d = 1$, $C_s(1) = 2\pi^{-s}\Gamma(s)\zeta(2s)$. Thus in two dimensions

$$\begin{aligned} \Gamma(s)\zeta(L_1, L_2; s) &= \Gamma(s) \sum_{n_1, n_2 > 0}{}' [\pi^2(n_1^2/L_1^2 + \ldots + n_d^2/L_d^2)]^{-s} \\ &= \tfrac{1}{4}\pi^{-s}[(L_1L_2)^s C_s(\omega_1, \omega_2) + [L_1^{2s} + L_2^{2s}]C_s(1)]. \end{aligned} \tag{5}$$

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According to (4), the first term in (5) has a pole at $s=1$ with residue $L_1 L_2/4\pi = \text{area}/4\pi$ and the second term has a pole at $s=\frac{1}{2}$ residue $-(L_2+L_2)/4\pi = -\text{perimeter}/8\pi$ (and both have poles at $s=0$, but this is due to the pole of $\Gamma(s)$). The generalisation to $d$ dimensions is obvious and $\Gamma(s)\zeta([L_1, \ldots, L_d; s)$ has poles at $s=d/2, d/2-1, \ldots,$ representing the volume, surface area, etc.

Itzykson *et al* are interested in the behaviour about $s=1$. For $s=1+\varepsilon$, (4) and (5) give $\zeta(L_1, L_2; s) = (L_1 L_2/4\pi)(1/\varepsilon + g + O(\varepsilon))$ with

$$g = \ln(L_1 L_2/4\pi) + \gamma + 1 + \sum_{n_1,n_2}{}' \exp(-\pi N^2)/(\pi N^2) + \sum_{n_1,n_2}{}' E_1(\pi \tilde{N}^2) - (\pi/3)(L_1/L_2 + L_2/L_1) \tag{6}$$

where we have used $C_1(1) = 2\pi^{-1}\zeta(2) = \pi/3$. This is the general form of $g$ for a rectangular domain.

For a perfect square, one can use the identity

$$\sum_{n_1,n_2}{}' (n_1^2+n_2^2)^{-s} = 4\zeta(s)\beta(s) \tag{7}$$

(where $\beta(s) = \Sigma(-1)^n(2n+1)^{-s}$) to find $C_s(1,1) = 4\Gamma(s)\pi^{-s}\zeta(s)\beta(s)$ and therefore $\zeta(L^2; s) = (L/\pi)^{-s}[\zeta(s)\beta(s) - \zeta(2s)]$. Then using the identity $C_s(1,1) = C_{1-s}(1,1)$ and the known behaviour of $\zeta(s)$ and $\beta(s)$ about $s=0$ (Erdelyi 1955, Abramowitz and Stegun 1965, Glasser 1973, Campbell and Ziff 1979),

$$\begin{aligned} \zeta(s) &= -\tfrac{1}{2} - (s/2)\ln(2\pi) + O(s^2) \\ \beta(s) &= \tfrac{1}{2} + s\ln[\Gamma^2(\tfrac{1}{4})2^{-1/2}\pi^{-1})] + O(s^2) \\ \Gamma(s) &= 1/s - \gamma + O(s) \end{aligned} \tag{8}$$

one can show

$$g = [\tfrac{1}{4}\ln(2^{1/2}L/\Gamma^2(\tfrac{1}{4})) + 2\gamma - 2\pi/3]. \tag{9}$$

We note that (9) is closely related to the energy of a two-dimensional square array of vortices in a rotating vessel (Campbell and Ziff 1979), and equivalently to the ground-state energy of a two-dimensional one-component plasma (which is a system of negatively charged parallel lines, in a uniform background of positive charge).

While the identity (7) seems to have been first found by Lorentz in 1871 (see Glasser and Zucker 1980), it has since been rediscovered by many people. The related identity they found for an equilateral triangle is also known. Glasser and Zucker (1980) have shown that a large number of rectangular and triangular systems of rational dimensions have energy levels whose sums can be written in a form like (7). Thus the degeneracy functions that Itzykson and Luck (1986) are investigating should be explicitly soluble fo a large number of geometries.

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